## Symmetric groups

In the last section, we showed how you can think of $D_{2n}$ as a set of some of the bijections from $\{1, 2, \ldots, 2n\}$ to itself. Of course, we need to preserve the geometric structure, so we don't include all bijections.

Let $\Omega$ be a set. Let $S_\Omega$ be the set of bijections from $\Omega$ to itself. Recall, $S_\Omega$ is a group w/ operation composition. It's called the symmetric group on $\Omega$.

When $\Omega = \{1, \ldots, n\}$, we denote $S_\Omega$ by just $S_n$.

Note: we can think of $S_n$ as the permutations of $\{1, \ldots, n\}$, so $|S_n| = n!$

## Cycle decomposition

Instead of describing elements of $S_n$ by listing where it sends $1, \ldots, n$, we have more efficient notation.

A cycle is a string of integers, written $(a_1 a_2 \ldots a_m)$, which represents the element of $S_n$ that sends $a_i$ to $a_{i+1}$ (for $i = 1, \ldots, m-1$), $a_m$ to $a_1$, and fixes the rest of $1, \ldots, n$.

**Ex:** $(1\ 3\ 2) \in S_4$ represents the permutation

$1 \mapsto 3$
$3 \mapsto 2$
$2 \mapsto 1$
$4 \mapsto 4$

We can write an arbitrary element of $S_n$ as a product of $k$ cycles $\underbrace{(a_1\ \dots\ a_{m_1})}_{\text{"length" } m_1}(a_{m_1+1}\ \dots\ a_{m_2})\dots(a_{m_{k-1}+1}\ \dots\ a_{m_k})$, called the cycle decomposition.

**Ex:** $(1\ 2)(3\ 4) \in S_4$ represents

$1 \mapsto 2$
$2 \mapsto 1$
$3 \mapsto 4$
$4 \mapsto 3$

In order to keep notation consistent, we need an algorithm to write each elt of $S_n$ as the product of disjoint cycles.

To demonstrate how this works, define $\sigma \in S_8$ by
$\sigma(1) = 5$, $\sigma(2) = 2$, $\sigma(3) = 6$, $\sigma(4) = 1$, $\sigma(5) = 4$, $\sigma(6) = 3$, $\sigma(7) = 8$, $\sigma(8) = 7$

## Cycle decomposition algorithm for elements of $S_n$

1.) Pick the smallest, $a$, of $\{1, 2, \dots, n\}$ which has not already appeared in a previous cycle. Begin the new cycle w/

a. If all $\{1, \ldots, n\}$ have appeared, go to step 4.

Ex: (1

2.) If $\sigma(a) = a$, close the cycle and go to step 1.
Otherwise the cycle continues w/ $b := \sigma(a)$

Ex: (1 5

3.) If $\sigma(b) = a$, close the cycle to complete it and go to step 1. Otherwise, continue the cycle w/ $c := \sigma(b)$. Repeat this step w/ $c$ as the new value for $b$ until the cycle closes.

Ex: $(1\ 5\ 4)(2)(3\ 6)(7\ 8)$

4.) Remove all cycles of length 1.

Ex: $(1\ 5\ 4)(3\ 6)(7\ 8)$

We'll come back to some details about this algorithm in a few weeks

## Multiplying elements of $S_n$

Since we treat elements of $S_n$ as functions, we multiply from right to left, so if $\sigma = (2\ 5\ 3)(1\ 4)$, $\tau = (1\ 2\ 4)$

Then $\sigma\tau = (1\ 5\ 3\ 2)$ and $\tau\sigma = (2\ 5\ 3\ 4)$.

**Claim**: Disjoint cycles in $S_n$ commute.

**Pf**: If $\sigma$ and $\tau$ are disjoint, and $i \in \{1, \dots, n\}$, then WLOG $\sigma(i) = i$. Thus, $\sigma(\tau(i)) \underset{\substack{\uparrow \\ \text{(disjointness)}}}{=} \tau(i) = \tau(\sigma(i))$. $\square$

**Note**: There are many ways to describe an element of $S_n$. e.g. $(12)(23) = (13)$, but we will prove later that there is a unique way to express each element as a product of disjoint cycles (up to reordering cycles, and cyclically permuting cycle elements), given by the output of the above algorithm.